Math 7110 – Homework 7 – Due: November 8, 2021

Practice Problems:

Problem 1. Dummit and Foote, section 7.4: 8, 9

Test practice:

Problem 2. Determine whether or not the following rings are fields:

(1) $\mathbb{Z}[x]/(x+1)$
(2) $\mathbb{R}[x]/(x^2-1)$
(3) $\mathbb{R}[x]/(x^2+1)$
(4) $\mathbb{Z}_2[x]/(x^2+x+1)$

Type solutions to the following problems in LaTeX, and email the tex and PDF files to me at `dbernstein1@tulane.edu` by 10am on the due date. Please title them as [lastname].tex and [lastname].pdf. When preparing your solutions, you must follow the rules as laid out in the course syllabus.

Graded Problems:

Problem 3. Assume R is commutative and let $I \subseteq R$ be an ideal. Define

$$\text{rad}(I) = \{r \in R : r^n \in I \text{ for some } n \geq 0\}.$$

An ideal is said to be *radical* if $I = \text{rad}(I)$.

(1) Prove that $\text{rad}(I)$ is an ideal of R
(2) Prove that prime ideals are radical
(3) Under what conditions on n is $(n) \subseteq \mathbb{Z}$ a radical ideal?

Problem 4. A *local ring* is a commutative ring with a unique maximal ideal. Let R be a local ring with maximal ideal M.

(1) Prove that every $f \in R \setminus M$ is a unit
(2) Prove that if R is commutative and has a 1, then if the set M of non-units forms an ideal, then R is local with maximal ideal M
(3) Let R be the ring of rational numbers with odd denominator. Prove that R is local with maximal ideal (2).